Frequentist Inference on Performance Measures of M/M/1 Queuing System

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Abstract: *In studying queuing systems, the main feature that reflects the system is the performance measures. Traffic intensity is considered one of the most important performance measures in the M/M/1 queuing system. In the present study we introduce a new frequents estimator for traffic intensity and examine its properties, and we also examine the sampling distribution of the estimator. We design a framework for testing hypotheses and construct confidence intervals and maximum likelihood estimators. We also performed a comparison with a few similar estimators of other authors.*
Keywords: *Queuing System, Traffic intensity, G3B distribution, Hyper geometric function, Euler Transformation.*

1. Introduction

The principle of queuing theory addresses the issue of overcrowding. It develops models to predict the behaviour of systems that strive to provide services to user groups. In this research work, we have taken the M/M/1 single-server system. The M/M/1 system receives customers one by one, forming a single queue for the entire system. For a queuing practitioner, the knowledge of the model's numerical values is critical. However, various factors such as time constraints and the need for accurate estimates can easily limit this knowledge. This chapter aims to provide a detailed discussion of these factors and their related procedures. In M/M/1, one of the most important performance measures is the traffic intensity denoted by ρ and expressed as the ratio of the average service time to the average inter-arrival time. As stated by Clarke (1957), a technique for approximating the distribution of the maximum queue length in a queuing system was presented. The paper established a mathematical platform for analysing queuing behaviour in a steady-state regime where one interest was the probability distribution of the queue length. Clarke's contributions provided an insight into the maximum queue size distribution, which has a great impact on systems performance, especially in conditions where capacity issues or limitation of resources have influence. This research attempts to bring to the field ideas concerning queuing length distribution in various practices, thereby contributing to the initial development of queuing theory. Muddapur (1972) reviewed methods of statistics to first, test for performance measures in queuing systems, and second, estimate the performance measures. This work concentrated on the M/M/1 queue model and analysed methods for obtaining an approximation of the queue size and time for waiting. Muddapur proposed and evaluated different assessment models to overcome problems related to the surety and effectiveness of the procedures. The paper offered significant knowledge enhancements for queuing system behaviour and allowed practical approaches for its performance measurement and analysis. Schruben and Kulkarni (1982) tried to establish the statistical properties of classical performance estimators for M/M/1 queueing systems. However, they were able to demonstrate that these simple estimators, like the mean queue length or the waiting time, have substantial drawbacks, mainly because they presented undefined SE. This phenomenon leads to unreliable and problematic estimations of performance measures in steady-state simulation. Previous work showed that there are such pathologic properties for queuing models and that more suitable estimation methods have to be sought in order to enhance the fidelity of performance predictions. McGrath and Singpurwalla (1987) conducted a comprehensive review of performance analysis and estimation in queues, utilizing Bayesian-based approaches. Their research was mostly focused on how prior knowledge was incorporated into sample information to improve estimation of the parameter as well as decision-making under a queuing system. They employed Bayesian methods to address issues such as reliability and the performance evaluation of system measures. The paper also compares and contrasts the functionalities of the two approaches, providing a brief overview of how helpful Bayesian inference is for analysing queuing systems. Basawa and Prabhu (1988) analysed stochastic processes for statistical inference, and this category includes queuing systems. Their work also addressed parametric and non-parametric approaches to estimation of the parameters of queues, especially concerning M/G/1 and other generalized queues. In particular, they stressed the role of the work with identifying proper and efficient estimators for these processes, some of the concerns being stationary, ergodicity, and the use of the likelihood function. They did enormous work on the theory of statistical inference in queuing models, providing easy methods of solving real-world problems. In the study, Armero and Bayari (1994) have also covered the applicability of statistical methods for both analysis and estimation of the parameters of the queuing system. Their major developments in their study were both to present and evaluate estimators for a number of performance indices, such as queues and waiting time in several queuing systems. They also provided approaches for how one can deal with issues that may arise with regard to bias and efficiency of the estimator when used in queuing systems that possess some level of complexity or are formulated in a non-conventional way. As stated in the paper, the identification of a proper method of estimation will improve the understanding of operation and control on queuing systems; thus, it has facilitated advancement in the field of queuing theory and application. Srinivas, Subba Rao, and Kale (2011) have mainly concentrated on the methods of statistical analysis for queuing. More precisely, the paper focused on estimating system parameters in queuing processes using simulation-based methods. They outlined methods to enhance the accuracy and reliability of estimators for queuing models, as well as the challenges in estimating model parameters when the system behaviour is complex or when the arrival distribution is unconventional. They brought new methods in simulation analysis, which provided ideas for solving practical issues in queuing systems, like evaluating performance and allocation of resources efficaciously. Chowdhury and Mukherjee (2013) have worked out maximum likelihood estimator (MLE) as well as Bayes estimator of traffic intensity in an $M/M/1/\infty$ queuing model in equilibrium based on supported range of number within the queue at ordered departure epochs. They conjointly derived estimates of some functions of traffic intensity which offer measures of effectiveness of the queue and a comprehensive simulation study beginning with the transition likelihood matrix.

To understand the future behaviour of queuing systems, Srinivas and Udupa (2014) presented an analytical comparison of simulation techniques to measure and predict the output from queuing systems. They concentrated on the heuristics generated in overhauling the performance estimators as a way of reducing variance in the system, especially within the queuing networks. They looked at ways to diminish the effects of the initial transient conditions and increase the accuracy of steady-state evaluations. Furthermore, the study focused on the issues of using simple arrival and service rates to model real-life systems where rates are random, different, and constant and put forward the strategies of improved system planning and resource management. This paper contributed to the improvement of simulation models necessary for queuing system analysis. Choudhury and Basak (2018) specifically focused on issues concerning queuing systems, with special emphasis on performance evaluation in systems that have non-orthodox arrival and service time processes. They looked at different queue arrangements and then imposed sophisticated stochastic methods to analyse and forecast the expected wait time, system percentage, and probability distribution of the queues. In the paper, the need to capture such realities as time-varying rates and unsteady service characteristics was well pointed out. They used the ideas, which improved the more general and real-life queuing models as well as extended queuing theories. Suyama, Quinino, and Cruz (2018) studied the performance and analysis of queuing systems using simulation-based research. Their work was centred around designing and testing ways in which the performance indices can be predicted with a high degree of precision given that they exist in demanding queuing systems. They classified different simulation methods for working as follows for evaluating factors such as bias and variance on performance estimators. The paper presented changes to simulation methodology to increase the credibility of the measures of the performance and gain deeper insight into the behaviour of the system. This work helped in getting nearer to the advancement of simulation analysis of queuing systems from both theoretical and practical points of view. Almeida et al. (2020) studied optimization, performance evaluation, and modelling for queuing systems through simulations and other statistical methods. Their study primarily focused on improving the reliability of performance indices, such as waiting time and the number of customers in the queue, by using innovative simulations. They attempted to determine how various parameters affect the system's performance and explored methods to increase the efficiency of queuing systems through optimal estimation and validation of the models. The paper also focused on real-life consequences and discussed recommendations for improving the layout of the system and distribution of resources when outcomes of the given simulation are considered. They contributed to the advancement of theoretical analysis and the practical implementation of solutions to enhance queuing systems. Das and Choudhury (2021) concentrated on determining a utilization factor for a power supply queuing model using the MLE and Bayesian methods. In the paper, one of the key characteristics, the utilization factor, was addressed in an attempt at creating estimators for it because it is essential to evaluating the capacity and availability of power supply systems. MLE and Bayesian estimators were compared in their efficiency; the authors looked at the effectiveness of both these methods. Most recently, Dutta and Choudhury (2023) derived some classical estimator of traffic intensity of M/M/1 queue system. In the present study, we introduce a new frequentist estimator for traffic intensity in the M/M/1 queuing system. We examined the sampling distribution of the estimator as well as its properties. Our estimator is appealing due to its desirable properties. We have shown how it can be applied to test hypotheses. We have constructed a confidence interval and a maximum likelihood estimator. A comparison is made with a few similar estimators.

2. Description of the M/M/1 model

In this model, we assume that the customers arriving at a queuing system are characterized by a Poisson distribution with a parameter λ. Because of the independent increment property of the Poisson process, the inter arrival times are independent random variables having an exponential distribution and the same rate λ. The system consists of the single server, while the service time for each customer is also independent and has an exponential distribution with the rate of μ. The number of customers is not restricted, and anyone can join the queue; the service is on a first-come, first-served basis. On the same note, the calling population has been assumed to be infinite.

The traffic intensity (ρ) is one of the most important parameters of the M/M/1 queuing model, which is defined as the ratio of the arrival rate (λ) to the service rate (μ). The traffic intensity must be less than one. "Assuming equilibrium is very frequent in queuing theory" by Armero and Bayarri (1999). If the traffic intensity exceeds one, it will lead to an uncontrollable situation in the queue length. Therefore, it is important that the condition $\lambda<\mu$ holds, which is necessary and sufficient for a stable queue. As the number of customers grows indefinitely, the restriction of $\rho<1$ prevents the queue length from exponentially increasing.

When the system fails to meet this condition, the operation manager(s) often make adjustments to prevent the queue from growing infinitely. This, in turn, guarantees that the restriction on traffic intensity is satisfied. To analyse queuing situations related to real life, different features are computed to check the effectiveness of the insight of the queuing system; these features are known as measures of performance of the queuing system. In general, there are three types of performance measures, viz., (i) number of customers currently in the waiting line or number of customers ongoing in the system, (ii) number of customers waiting in the system or queue, and (iii) an identifier of the server's state or time during which the server may not be useful. "Since most queuing systems have stochastic elements, these measures are often random variables, so their probability distributions—or at least their expected values—are sought" by Shortle et al. (2018). The following are the widely used measures of performance in the M/M/1 model:

(i) Average system size, $L_s = \frac{\rho}{1-\rho}$

(ii) Average queue size, $L_q = \frac{\rho^2}{1-\rho}$

(iii) Average waiting time in the system, $W_s = \frac{1}{\mu(1-\rho)}$

(iv) Average waiting time in the queue, $W_q = \frac{\rho}{\mu(1-\rho)}$

(v) Traffic intensity, $\rho = \frac{\lambda}{\mu}$

3. **Derivation of a statistic and its distribution:**

Consider that we are observing the two data sets: The model incorporates two separate environments for the two processes; the arrival process and the service process. The random variable x depicts the interarrival time with an exponential distribution and the rate parameter given by λ, while the random variable y depicts the service time with an exponential distribution and the rate parameter given by μ. Let a random sample $x_1, x_2, \ldots, x_m$ of m inter-arrival times have been taken from the distribution of x, and a random sample $y_1, y, \ldots, y_n$ of n service times have been taken from the distribution of y.

Now, let us define $u_1 = \sum_{i=1}^{m} x_i$ and $u_2 = \sum_{j=1}^{n} y_j$ then $u_1 \sim \gamma\,(m, \lambda)$ and $u_2 \sim \gamma\,(n, \mu)$, by Feller (1950)

Due to the independence of inter-arrival times from service times, we assume that the two samples are independent of each other, and hence u_1 and u_2 are also independent.

The joint pdf of u_1 and u_2 is as follows

$$f_1(u_1, u_2) = \frac{\lambda^m}{\Gamma m} e^{-\lambda u_1} {u_1}^{m-1} \frac{\mu^n}{\Gamma n} e^{-\mu u_2} {u_2}^{n-1}, \quad u_1 \geq 0, u_2 \geq 0$$

Now, consider the statistic, $w = \frac{u_1}{u_1 + u_2}$ and $z = (u_1 + u_2)$

Now, Jacobian of the transformation is given by

$$J = \begin{vmatrix} \frac{\partial u_1}{\partial w} & \frac{\partial u_2}{\partial w} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} \end{vmatrix} = z$$

$$|J| = z$$

Therefore, the joint pdf of w and z is given by

$$f_2(w, z) = \frac{\lambda^m}{\Gamma m} e^{-\lambda wz} (wz)^{m-1} \frac{\mu^n}{\Gamma n} e^{-\mu(1-w)z} [(1-w)z]^{n-1} z$$

$$= \frac{\lambda^m}{\Gamma m} \frac{\mu^n}{\Gamma n} e^{-\left[\frac{\lambda}{\mu} w + (1-w)\right]\mu z} w^{m-1} (1-w)^{n-1} z^{m+n-1}, \quad 0 \leq w \leq 1, z \geq 0 \quad (1)$$

Integrating equation (1) with respect to z, we get

$$f(w) = \frac{\lambda^m}{\Gamma m} \frac{\mu^n}{\Gamma n} \frac{w^{m-1}(1-w)^{n-1}}{\left[\frac{\lambda}{\mu} w + (1-w)\right]^{m+n} \mu^{m+n}}$$

$$= \frac{\rho^m w^{m-1}(1-w)^{n-1}}{\beta(m,n)(1-(1-\rho)w)^{m+n}}, \quad 0 \leq w \leq 1, \ 0 < \rho \left(= \frac{\lambda}{\mu}\right) < 1 \qquad (2)$$

which is the $G3B(m, n, \rho)$ distribution by Chen and Novick (1984)

4. An expression

For $G3B(m,n,\rho)$, we determine the followings

$$E(W^r) = \frac{\rho^m}{\beta(m,n)}\int_0^1 \frac{w^{m+r-1}(1-w)^{n-1}}{(1-(1-\rho)w)^{m+n}}dw,\ m+r>0$$

$$= \frac{\rho^m}{\beta(m,n)}\beta(m+r,n)_2F_1(m+r,m+n;m+n+r;1-\rho) \qquad (3)$$

where $_2F_1(b,c;d;\delta) = \frac{1}{\beta(b,d-b)}\int_0^1 \frac{z^{b-1}(1-z)^{b-d-1}}{(1-\delta z)^c}dz$ is the Euler integral for hypergeometric function by Exton (1976).

"The Euler integral is unchanged by the transformations" by Exton (1976)

Therefore, by applying Euler transformations

$2F1(b,c;d;\delta) = (1-\delta)^{d-b-c}{}_2F_1(d-b,d-c;d;\delta)$by Chen and Novick (1984), we get

$$E(W^r) = \frac{\Gamma(m+n)\Gamma(m+r)}{\Gamma(m)\Gamma(m+n+r)}{}_2F_1(n,r;m+n+r;1-\rho) \text{ for } \rho<1 \qquad (4)$$

For large m & n by Chen and Novick (1984)

$_2F_1(n,r;m+n+r;1-\rho)$ approaches to$\left(\frac{m+n}{m+n\rho}\right)^r$

(5)

Therefore, using equation (4)& (5),we obtain

$$E(W^r) = \frac{\Gamma(m+n)\Gamma(m+r)}{\Gamma(m)\Gamma(m+n+r)}\left(\frac{m+n}{m+n\rho}\right)^r \qquad (6)$$

In particular, taking $r=1$, we get

the mean of the G3B distribution is $\frac{1}{1+\frac{n}{m}\rho}$.

Now, an unbiased estimator can be constructed forρ.

Again, taking$r=-1$ in (6), we get

$$E\left(\frac{1}{W}\right) = \frac{(m+n-1)(m+n\rho)}{(m-1)(m+n)}$$

which give us $E\left(\frac{(m-1)(m+n)}{n(m+n-1)W} - \frac{m}{n}\right) = \rho$

Therefore, $\left[\frac{(m-1)(m+n)}{n(m+n-1)W} - \frac{m}{n}\right]$ is an unbiased estimator of ρ.

(7)

Again,taking$r=-2$ in (6), after some simple algebra, we obtain

$$Var\left(\frac{(m-1)(m+n)}{n(m+n-1)W} - \frac{m}{n}\right) = \frac{(m+n\rho)^2}{n(m+n-1)(m-2)}$$

Now, $Var\left(\frac{(m-1)(m+n)}{n(m+n-1)W} - \frac{m}{n}\right) \to 0\ as\ m\to\infty, n\to\infty$

which shows that $\left[\frac{(m-1)(m+n)}{n(m+n-1)W} - \frac{m}{n}\right]$ is a consistent estimator of ρ.

Again from (2), we have

$$f(w) = \frac{\rho^m w^{m-1}(1-w)^{n-1}}{\beta(m,n)(1-(1-\rho)w)^{m+n}}, 0\le w\le 1$$

$$= g(t(w),\rho)h(w)$$

Consequently, by applying the Fisher-Neyman Factorization theorem, t(w) is identified as a sufficient estimator for ρ, namely, $\left[\frac{(m-1)(m+n)}{n(m+n-1)W} - \frac{m}{n}\right]$functions as a sufficient estimate of ρ.

Our estimator is attractive due to its impartial nature, consistency, and sufficiency. It satisfies three characteristics of an effective estimator.We have noticed that our estimator is appealing, as it is unbiased as well as consistent and sufficient. That is, it fulfils three properties of a good estimator.

5. Testing of Hypothesis of ρ

Here, we now develop a procedure for testing the hypothesis aboutρ.

For this purpose, we use the probability density function from equation (2) as given below:

$$f(w) = \frac{\rho^m w^{m-1}(1-w)^{n-1}}{\beta(m,n)(1-(1-\rho)w)^{m+n}}, 0 \le w \le 1$$

Let $w = \frac{m}{m+nt}$

$$\therefore f(t) = \frac{\rho^m \frac{n}{m}\left(\frac{m}{m+nt}\right)^{m+1}\left(1-\frac{m}{m+nt}\right)^{n-1}}{\beta(m,n)(1-(1-\rho)\frac{m}{m+nt})^{m+n}}, t \ge 0 \qquad (8)$$

Now, we test$H_0: \rho = \rho_0$ against$H_1: \rho = \rho_1$

Let us suppose that we try to develop an α-level critical region by applying NP lemma (Neyman-Pearson), the most powerful critical region (MPCR), given by

$$\frac{f(t,\rho_1)}{f(t,\rho_0)} > c\ (say)$$

$$=> \left(\frac{\rho_1}{\rho_0}\right)^m \left(\frac{nt+m\rho_0}{nt+m\rho_1}\right)^{m+n} > c, \text{using equation (8)}$$

$$=> \frac{nt+m\rho_0}{nt+m\rho_1} > \left(\frac{c}{\left(\frac{\rho_1}{\rho_0}\right)^m}\right)^{-(m+n)} = c_1(\text{say})$$

Case I. If $\rho_1 > \rho_0$ then

$$nt + m\rho_0 > c_1(nt + m\rho_1) => t > \frac{m(c_1\rho_1 - \rho_0)}{n(1-c_1)} = k_1(say)$$

Thus, the critical region is $T = (t: t > k_1)$

The constants k_1 are chosen in such a way that the size of the critical region equals α.

i.e., $P(t \in T|H_0) = \alpha => P_{H_0}(t > k_1) = \alpha$

$$=> \int_{k_1}^{\infty} \frac{\rho_0^m \frac{n}{m}\left(\frac{m}{m+nt}\right)^{m+1}\left(1-\frac{m}{m+nt}\right)^{n-1}}{\beta(m,n)(1-(1-\rho_0)\frac{m}{m+nt})^{m+n}} dt = \alpha$$

$$=> \int_{k_1}^{\infty} \frac{\left(\frac{n}{m}\right)^n t^{n-1}}{\beta(m,n)\rho_0^n(1+\frac{nt}{m\rho_0})^{m+n}} dt = \alpha$$

Taking $\frac{nt}{m\rho_0} = s$, we get

$=> \int_{\frac{nk_1}{m\rho_0}}^{\infty} \frac{s^{n-1}}{\beta(m,n)(1+s)^{m+n}} dt = \alpha$ (9)

an incomplete beta integral, and one can solve it byMazumdar and Bhattacharjee (1973)

Case II. If $\rho_1 < \rho_0$ then

$$nt + m\rho_0 < c_2(nt + m\rho_1) => t < k_2(say)$$

Thus, the critical region is $T_1 = (t: t < k_2)$

Similarly, one can calculated $k_2$from (10) as given below by Mazumdar and Bhattacharjee (1973).

$\int_0^{\frac{nk_2}{m\rho_0}} \frac{s^{n-1}}{\beta(m,n)(1+s)^{m+n}} dt = \alpha$ (10)

6. Confidence interval of ρ

Let γ_1 *and* γ_2 be the lower and the upper limits, respectively, for the estimator given in (7);then γ_1 *and* γ_2 can be determined by using the size condition.The equations for the purpose aregiven below:

$$P(t < \gamma_1) = \frac{\alpha}{2} \; and \; P(t > \gamma_2) = \frac{\alpha}{2}$$

Now, $P(t < \gamma_1) = \frac{\alpha}{2} => \int_0^{\gamma_1} \frac{\rho_0^m \frac{n}{m}\left(\frac{m}{m+nt}\right)^{m+1}\left(1-\frac{m}{m+nt}\right)^{n-1}}{\beta(m,n)(1-(1-\rho_0)\frac{m}{m+nt})^{m+n}} dt = \frac{\alpha}{2}$

$=> \int_0^{\frac{n\gamma_1}{m\rho_0}} \frac{s^{n-1}}{\beta(m,n)(1+s)^{m+n}} dt = \frac{\alpha}{2}$ (11)

and $P(t > \gamma_2) = \frac{\alpha}{2} => \int_{\gamma_2}^{\infty} \frac{\rho_0^m \frac{n}{m}\left(\frac{m}{m+nt}\right)^{m+1}\left(1-\frac{m}{m+nt}\right)^{n-1}}{\beta(m,n)(1-(1-\rho_0)\frac{m}{m+nt})^{m+n}} dt = \frac{\alpha}{2}$

$=> \int_{\frac{n\gamma_2}{m\rho_0}}^{\infty} \frac{s^{n-1}}{\beta(m,n)(1+s)^{m+n}} dt = \frac{\alpha}{2}$ (12)

The incomplete beta integrals (11) and (12) can be solved by Mazumdar and Bhattacharjee (1973).

7. Maximum likelihood estimator (MLE) of ρ

Given the $G3B(m,n,\rho)$ distribution, for a sample $\{w_1, w_2, \dots, w_k\}$ the likelihood function is

$$L = \prod_{l=1}^{k} f(w_l) = \{\beta(m,n)\}^{-k} \rho^{mk} \prod_{l=1}^{k} \left[w_l^{m-1}(1-w_l)^{n-1}(1-(1-\rho)w_l)^{-(m+n)}\right]$$

$\therefore logL = -klog\{\beta(m,n)\} + mklog(\rho) + (m-1)\sum_{l=1}^{k}\log(w_l) + (n-1)\sum_{l=1}^{k}\log(1 - w_l) - (m+n)\sum_{l=1}^{k}\log(1-(1-\rho)w_l)$ (13)

Differentiating (13) with respect toρ, and then equating to zero, we get

$$\frac{km}{\rho} - (m+n)\sum_{l=1}^{k}\frac{w_l}{(1-[1-\rho]w_l)} = 0$$

$$=> \frac{km}{(m+n)\rho} = \sum_{l=1}^{k} w_l[1-(1-\rho)w_l]^{-1}$$

$=> \frac{km}{(m+n)\rho} = \sum_{l=1}^{k} w_l\,[1+(1-\rho)w_l]$, ignoring higher order,

since $0 \le w_l \le 1$ $\quad and\ 0 < (1-\rho) < 1.$

$$=> \left(\sum_{l=1}^{k} w_l^2\right)\rho^2 - \left(\sum_{l=1}^{k}(w_l + w_l^2)\right)\rho + \frac{km}{(m+n)} = 0$$

$$=> \rho = \frac{\left[\sum_{l=1}^{k}(w_l + w_l^2) \pm \sqrt{\left(\left(\sum_{l=1}^{k}(w_l+w_l^2)\right)^2 - 4\frac{km}{(m+n)}\left(\sum_{l=1}^{k} w_l^2\right)\right)}\right]}{2\left(\sum_{l=1}^{k} w_l^2\right)}$$

Since $0 \le w_l \le 1$ and $0 < \rho < 1$, therefore,

$$\rho = \frac{\left[\sum_{l=1}^{k}(w_l+w_l^2) - \sqrt{\left(\left(\sum_{l=1}^{k}(w_l+w_l^2)\right)^2 - 4\frac{km}{(m+n)}\left(\sum_{l=1}^{k} w_l^2\right)\right)}\right]}{2\left(\sum_{l=1}^{k} w_l^2\right)}$$

Thus, the MLEforρ is given by

$$\hat{\rho} = \frac{\left[\sum_{l=1}^{k}(w_l+w_l^2) - \sqrt{\left(\left(\sum_{l=1}^{k}(w_l+w_l^2)\right)^2 - 4\frac{km}{(m+n)}\left(\sum_{l=1}^{k} w_l^2\right)\right)}\right]}{2\left(\sum_{l=1}^{k} w_l^2\right)} \qquad (14)$$

8. Simulations

Simulation is done using R programming and simulated 5000 times.

Table 1: Estimates of ρ using unbiased estimator with MSE (in parenthesis) for different sample sizes.

True value of ρ	Size of the sample				
	30	50	100	150	250
$\rho = 0.5$ MSE	0.4920939 (0.0185339)	0.4951609 (0.01045644)	0.4965752 (0.005084875)	0.4983305 (0.003357661)	0.4996641 (0.001927169)
$\rho = 0.7$ MSE	0.6957111 (0.03622232)	0.6972657 (0.0204562)	0.6972154 (0.009951119)	0.6990005 (0.006576552)	0.7003314 (0.003777141)
$\rho = 0.9$ MSE	0.8993283 (0.05984775)	0.8993705 (0.03380339)	0.8978555 (0.01644159)	0.8996704 (0.0108699)	0.9009986 (0.006244661)

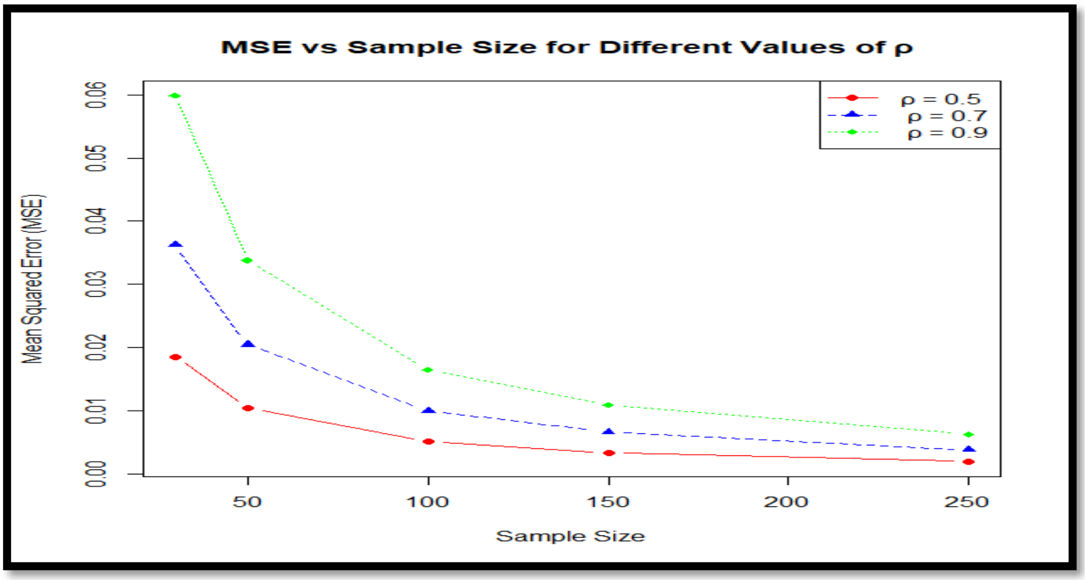


Figure 1. MSE vs Sample sizes for different values of ρ=0.5, 0.7, 0.9 using unbiased estimator of ρ.

Table 2: Estimates of ρ using maximum likelihood estimator with MSE (in parenthesis) for different sample sizes.

True value of ρ	Size of the sample			
	30	50	100	150
$\rho = 0.5$ MSE	0.5002767 (0.00006806055)	0.5000227 (0.00004045236)	0.5000617 (0.00001991883)	0.5000147 (0.00001356291)
$\rho = 0.7$ MSE	0.7003873 (0.0001334015)	0.7000314 (0.00007928843)	0.7000861 (0.00003904228)	0.7000202 (0.00002658417)
$\rho = 0.9$ MSE	0.9004986 (0.0002205217)	0.9000410 (0.0001310686)	0.9001114 (0.00006453939)	0.9000267 (0.00004394526)

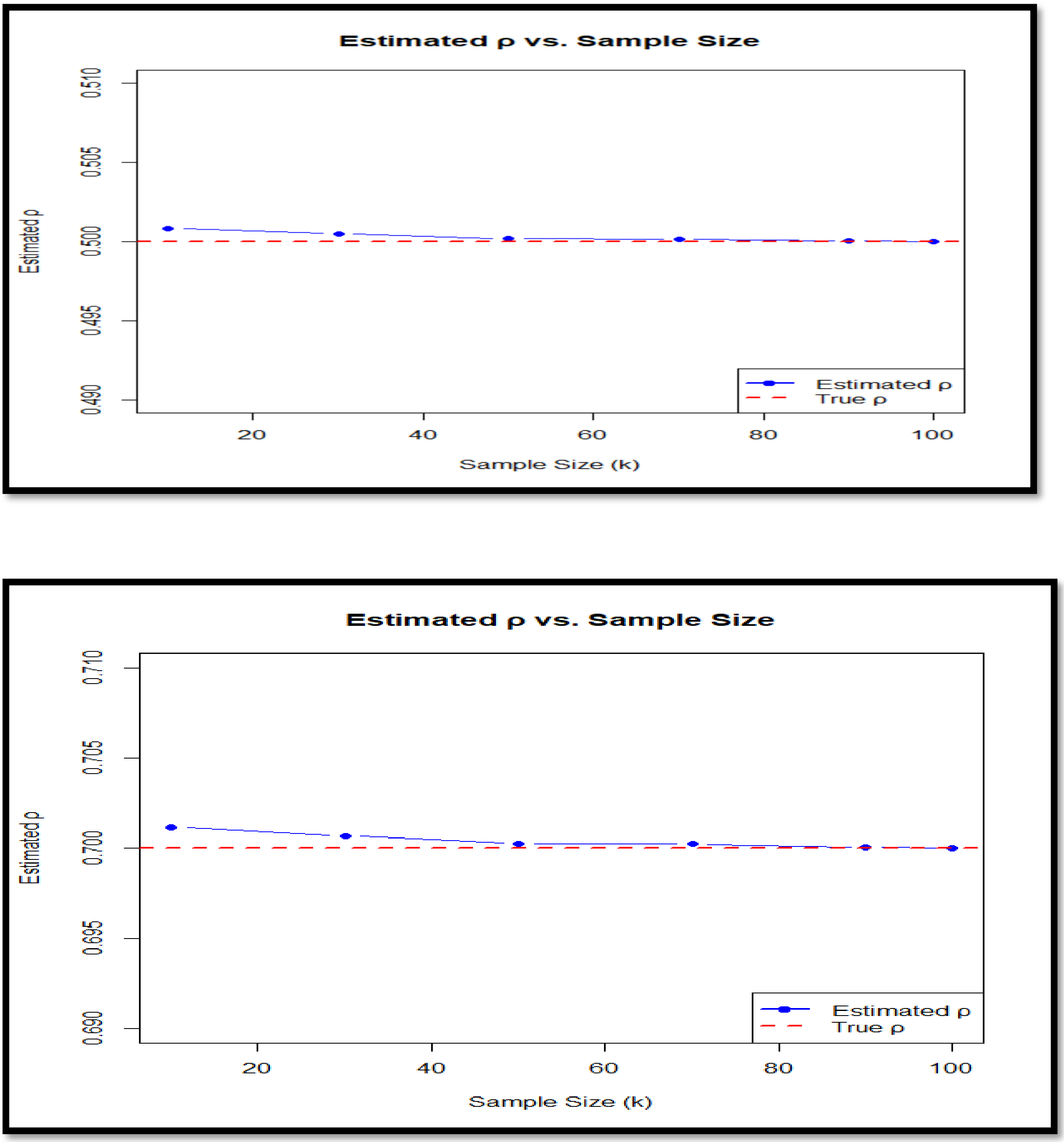
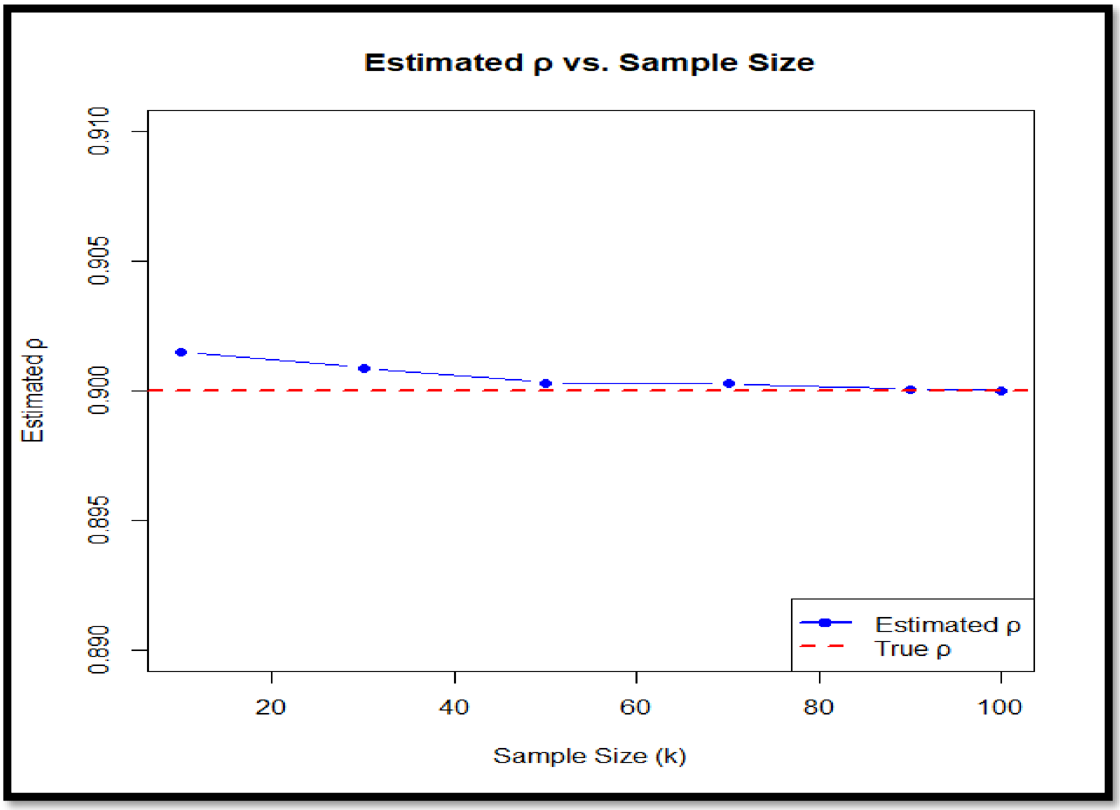


Figure 2. Estimated value of ρ using ML estimator vs sample size for ρ=0.5, 0.7


Estimated ρ vs. Sample Size
Estimated ρ
0.890
0.895
0.900
0.905
0.910
20
40
60
80
100
Sample Size (k)
Estimated ρ
True ρ

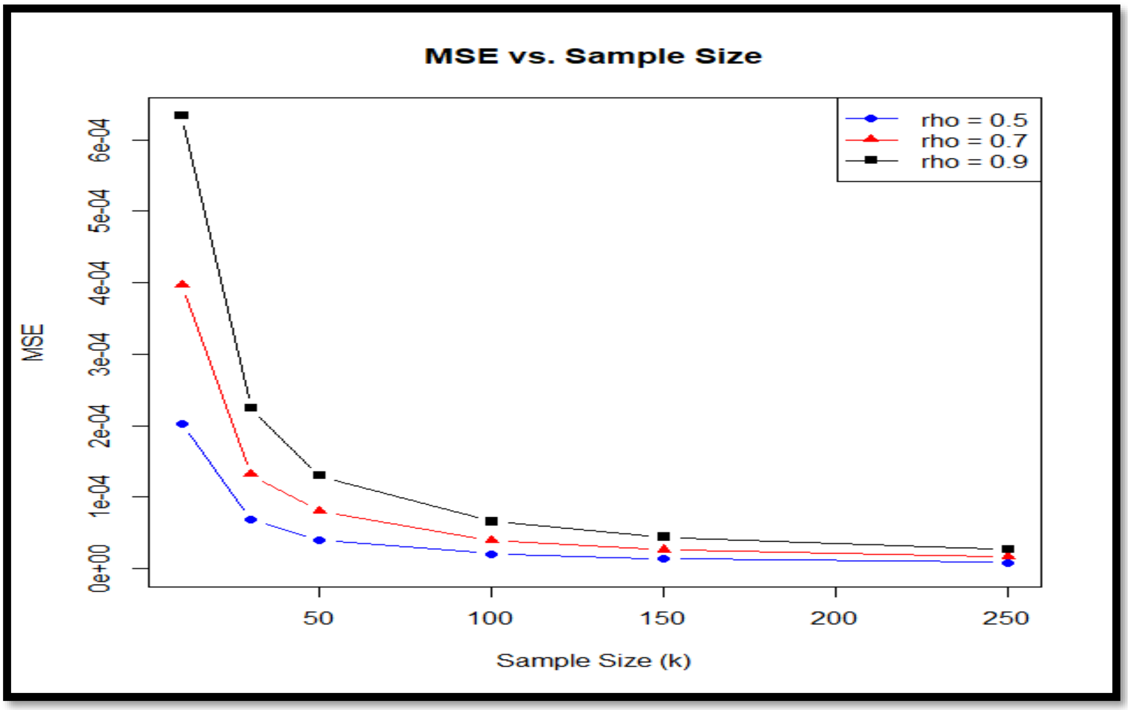

MSE vs. Sample Size
MSE
0e+00
1e-04
2e-04
3e-04
4e-04
5e-04
6e-04
50
100
150
200
250
Sample Size (k)
rho = 0.5
rho = 0.7
rho = 0.9

Figure 3. Estimated value of ρ using maximum likelihood estimator vs sample size for ρ=0.9, and MSE vs Sample size for different values of ρ=0.5, 0.7, 0.9 using ML estimator of ρ.

From Table 1, Table 2, Figure 1, Figure 2 and Figure 3, it is seen that as the sample size increases unbiased estimates and ML estimates of ρ improves and mean square errors decreases.

9. Comparisons of our ML estimator with others

We compared our maximum likelihood estimator with the ML estimator by Chowdhury and Mukherjee (2013). The ML estimator of ρ by Chowdhury and Mukherjee (2013) is given by

$$\hat{\rho}_{mle} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

where $A = N + n_{00} + n_{10} - n_0 - 1$

$$B = N + \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty}(j-i-1)n_{ij} + n_{00} + n_{10} + \sum_{j=0}^{\infty} j\left(n_{0j} + n_{1j}\right) + 1$$

$$C = n_0 + \sum_{j=0}^{\infty} j\left(n_{0j} + n_{1j}\right) + \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty}(j-i-1)\, n_{ij}$$

$$N = \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty} n_{ij}\,, n_{00} = \sum_{j=0}^{\infty} n_{0j}\,, n_{10} = \sum_{j=0}^{\infty} n_{1j}$$

and n_{ij}, the observed number of transitions from state i to state j in N_t.

For comparison, we followed the same simulation procedure that was outlined by Chowdhury and Mukherjee (2013) and simulated 1000 times to get RMSE. Estimated values of our maximum likelihood estimator are given in Table 3

Table 3: Estimates of ρ using maximum likelihood estimator with RMSE (in parenthesis) for sample sizes (30, 50, 100)

True value of ρ	Sample size					
	Our estimator			Estimator of Chowdhury and Mukherjee		
	30	50	100	30	50	100
$\rho = 0.5$ RMSE	0.5006402 (0.008063451)	0.5001845 (0.006416161)	0.5001406 (0.004554429)	0.4823 (0.0234151)	0.4941 (0.0169929)	0.4956 (0.0102145)
$\rho = 0.7$ RMSE	0.7008962 (0.011288959)	0.7002579 (0.008982706)	0.7001966 (0.006376323)	0.6739 (0.0314587)	0.6812 (0.0214703)	0.6998 (0.0154366)
$\rho = 0.9$ RMSE	0.9011529 (0.01451443)	0.9003322 (0.011549211)	0.9002534 (0.008198148)	0.8688 (0.0245871)	0.8809 (0.0124587)	0.8912 (0.0095487)

Here, we compared our MLE of ρ with the MLE of ρ by Basak and Choudhury (2018). They obtained the MLE of ρ as given below.

$$\hat{\rho}_{ML} = \frac{\sqrt{n_2^2 + 4(y+2n)(y+n_2)} - n_2}{2(y+2n)}$$

where $y = \sum_{i=1}^{n_2} x_i$, n the observed number of M/M/1 queues, x_i non-empty queue size of n_2 observations. We adhered strictly to the technique defined by Basak and Choudhury (2018). The simulation procedure was repeated 5000 times to MSE, MSE as carried out by Basak and Choudhury (2018).

Table 4: Estimates of ρ using maximum likelihood estimator with MSE (in parenthesis) for sample sizes (30, 50, 100) by(14)

True value of ρ	Sample size					
	Our Estimator			Estimator of Basak and Choudhury		
	30	50	100	30	50	100
$\rho = 0.5$ MSE	0.5002356 (0.000103)	0.5000227 (0.0000405)	0.5000617 (0.0000199)	0.480953 (0.008773)	0.492379 (0.00295)	0.496542 (0.001402)
$\rho = 0.8$ MSE	0.800377 (0.000263)	0.800036 (0.000104)	0.8000985 (0.0000510)	0.789904 (0.00229)	0.795285 (0.000815)	0.797708 (0.000350)
$\rho = 0.9$ MSE	0.9004245 (0.000333)	0.900041 (0.000131)	0.9001114 (0.0000645)	0.887786 (0.000698)	0.890558 (0.000276)	0.892443 (0.000169)

From Tables 3 and 4, it is evident that our maximum likelihood estimator provides better estimates compared to those of Chowdhury and Mukherjee (2013) and Basak and Choudhury (2018). Additionally, as the sample size increases, the RMSE decreases more significantly than in Chowdhury and Mukherjee (2013), and the MSE decreases more substantially than in Basak and Choudhury (2018). Based on these results we conclude that our maximum likelihood estimator provides estimates of traffic intensity that are closest to the actual values, making it superior to those available in the literature. This highlights a key advantage of our estimator.

10. Conclusions

We presented two estimators for the traffic intensity in an M/M/1 queuing system: Two of the estimators include an unbiased estimator and a maximum likelihood (ML) estimator. It can be seen that the unbiased estimator possesses some attributes that are characteristic of a good estimator. A confidence interval was built, and the most powerful critical region for the estimator of traffic intensity was developed. The simulation procedure was adopted in order to compare the newly developed ML estimator of traffic intensity with those existing in the literature, where it was revealed that the new estimator performs better than the existing ones.

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